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PROPERTIES OF FOUR CONFOCAL PARABOLAS WHOSE VERTICAL TANGENTS FORM A SQUARE.

BY C. M. HEBBERT.

In a recent paper* I have shown that if the diagonals of a quadrilateral are perpendicular and equal, an infinite number of squares may be circumscribed to the quadrilateral, and that if two squares can be inscribed in a given quadrilateral, an infinite number of squares may be inscribed in it. In an article in the Bulletin de l'Academie Imperiale des Sciences de St. Petersbourg, vol. VII, p. 177, Thomas Clausen pointed out an apparent, though only apparent, dualism existing between the squares in these two cases, but was unable to give a satisfactory explanation. This paper shows that the squares inscribed in the quadrilateral are a single infinity of the ∞^2 squares circumscribed to an infinite set of S-quadrangles, i. e., quadrangles whose diagonals are perpendicular and equal.

The equations of four confocal parabolas, which we call G-parabolas, whose vertical tangents form a square, may be written in the form

$$(G_1)\quad y^2 = 4px + 4p^2, \qquad (G_2)\quad y^2 = 4(p-c)x + 4(p-c)^2,$$

$$(G_3)\quad x^2 = 4q^2 - 4qy, \qquad (G_4)\quad x^2 = 4(c-q)y + 4(c-q)^2,$$

where c is the side of the square formed by the vertical tangents and the common focus is at the origin F (see figure).

The equations of tangents to G_1 and G_2 having the same slope m are

$$(t_1)\quad y = mx + \frac{m^2+1}{m}\,p, \qquad \text{and} \qquad (t_2)\quad y = mx + \frac{m^2+1}{m}\,(p-c).$$

The equations of tangents to G_3 and G_4 which are perpendicular to t_1 and t_2 are

$$(t_3)\quad y = -\frac{1}{m}x + q\frac{m^2+1}{m^2}, \qquad \text{and} \qquad (t_4)\quad y = -\frac{1}{m}x + \frac{m^2+1}{m^2}(q-c).$$

Inspection of equations (t_1) and (t_2) shows that the distance between them is equal to the distance between the lines (t_3) and (t_4). Hence we have

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* "The inscribed and circumscribed squares of a quadrilateral and their significance in kinematic geometry," *Annals of Math.*, Vol. 16, p. 38. In the present paper this is referred to as "the preceding paper." By an unfortunate oversight references to the work of Thomas Clausen, referred to in this paper, and Carnot, Geometrie de Position, p. 374, were omitted in that article.

Theorem I. *Any four rectangular tangents to this system of parabolas form a square.*

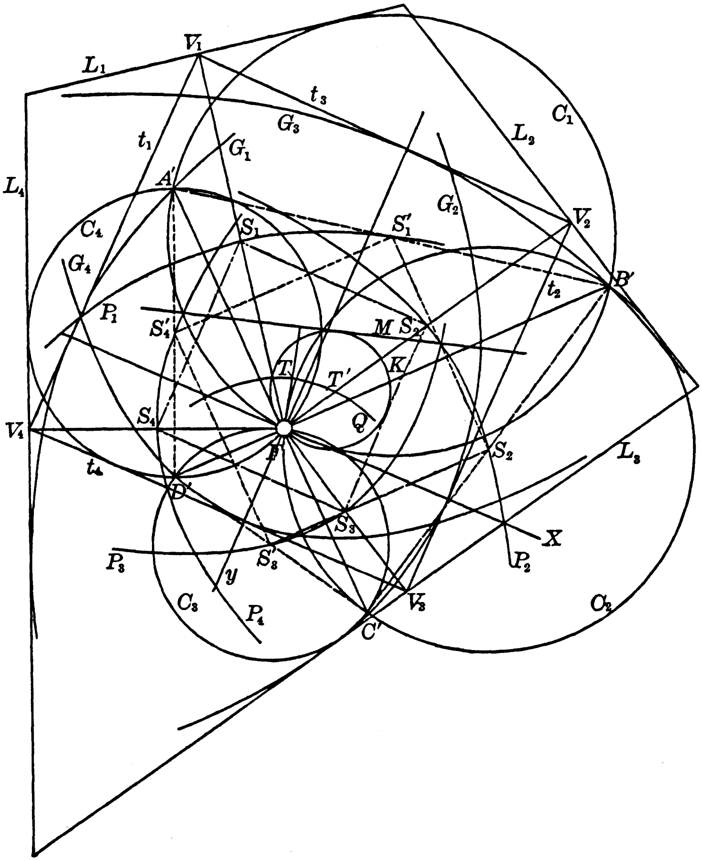
The sect-point of t_1 and t_3 is the point $(q/m - p, q + p/m)$, which lies on the line $p(p + x) = q(y - q)$, which is designated by L_1.

Loci of sect-points (t_2, t_3), (t_2, t_4), (t_1, t_4), respectively, are

$$(p - c)(x + p - c) = q(y - q), \tag{L_2}$$

$$(p - c)(x + p - c) = (q - c)(y - (q - c)), \tag{L_3}$$

$$p(x + p) = (q - c)(y - (q - c)). \tag{L_4}$$



These are tangents to G_1, G_2, G_3, G_4, in pairs, i. e., each of the lines L_1, L_2, L_3, L_4, is tangent to two of the parabolas and each of the parabolas is tangent to two of the lines.

The coördinates of the center of the square formed by the four tangents are

$$\left(\frac{2q + (m - 1)c}{2m} - p, \quad q - \frac{(m + 1)c - 2p}{2m}\right);$$

and the locus of the center is, therefore, the line

$$(M) \qquad (2y - (2q - c))(2q - c) = (2p - c)(2x + 2p - c).$$

THEOREM II. *The vertices of the variable square formed by the rectangular tangents of Theorem I always lie on straight lines which are tangent to the parabolas in pairs. The center of the variable square lies on a fixed straight line.*

The results of this paper thus far, if considered in connection with the results of the preceding paper, show that there is a (1, 1) correspondence between points of the parabolas and points of the sides of the quadrilateral whose inscribed squares envelope these parabolas. For, in the preceding paper we saw that the quadrilateral uniquely determined the four confocal parabolas while in this section we have shown that the four confocal parabolas uniquely determine a quadrilateral. To every point on a side of the quadrilateral corresponds a side of an inscribed square and thence a point of one of the parabolas, and conversely. We shall use this property instead of carrying out some extended analytic processes in what follows.

Any line, $y = nx$, through the focus, cuts the parabolas G_1 and G_2 in the points

$$A'\colon \quad \left(\frac{2p \pm 2p\sqrt{1 + n^2}}{n^2}, \quad \frac{2p \pm 2p\sqrt{1 + n^2}}{n}\right),$$

and

$$C'\colon \quad \left(\frac{2(p - c) \pm 2(p - c)\sqrt{1 + n^2}}{n^2}, \quad \frac{2(p - c) \pm 2(p - c)\sqrt{1 + n^2}}{n}\right),$$

respectively.

The length of the segment cut out of the line by these two points is $d_1 = 2c(1 \pm \sqrt{1 + n^2})\sqrt{1 + n^2}/n^2$.

The line $y = -x/n$, perpendicular to $y = nx$, cuts G_3 and G_4 in the points B' $(2q \pm 2q\sqrt{1 + n^2})/n$, $(-2q \pm 2q\sqrt{1 + n^2})/n^2)$ and D' $((2(q - c) \pm 2(q - c)\sqrt{1 + n^2})/n$, $(-2(q - c) \pm 2(q - c)\sqrt{1 + n^2})/n^2)$ respectively, and the length of their join is $d_2 = d_1$, above.

From this and Theorem IV, Corollary 1, of the preceding paper follows

THEOREM III. *Any two perpendicular lines through the focus of this system of parabolas cut the two opposite pairs of this system in four points forming an S-quadrangle, i. e., a quadrangle about which may be circumscribed an infinite number of squares.*

Since the coördinates of these four points in the same order satisfy the four equations t_1, t_2, t_3, t_4, simultaneously, they must be the points of contact of a set of rectangular tangents and n is a function of m: $n(1 - m^2) = 2m$.

Now for a given value of n, the vertices of the infinite set of squares belonging to the S-quadrangle so determined lie on four fixed circles, C_1, C_2, C_3, C_4. These four circles must be tangent to the four lines, L_1, L_2, L_3, L_4, on which the vertices of the squares circumscribed to the four G-parabolas lie. For, if the circles cut the lines, there would be two squares having the same points of contact with the G-parabolas and having their vertices on the four fixed lines. This is impossible since there is a (1, 1) correspondence between the points of these four lines and the four parabolas.

The same reasoning applies to the locus of centers of squares corresponding to given values of n, i. e., the circle K which is the locus of the centers of squares passing through these four points is tangent to the line M on which lie the centers of all the squares tangent to the system of parabolas. Analytically, the proof is as follows:

The circle, K, is determined by the focus F and the mid-points of the segments cut out of $y = nx$ and $y = -x/n$ by the parabolas, i. e., the mid-points of $A'C'$ and $B'D'$. The equation of this circle K is

$$x^2 - \frac{(2p + 2nq - c - cn)(1 \pm \sqrt{1 + n^2})}{n^2} x + y^2 - \frac{(c - 2q + 2pn - cn)(1 \pm \sqrt{1 + n^2})}{n^2} y = 0.$$

The equation of M, which we have determined, may be written

$$y = \frac{(4p - 2c)x + (2p - c)^2 + (2q - c)^2}{4q - 2c}.$$

The sect-point of M and K, determined by substituting this value of y in the equation of K and reducing, has an abscissa given by the value of x in the equation

$$4n^2x^2 + 4n[n(2p - c) - (1 \pm \sqrt{1 + n^2})(2q - c)]x + [n(2p - c) - (1 \pm \sqrt{1 + n^2})(2q - c)]^2 = 0.$$

Since this expression is a perfect square and M is not parallel to the y-axis, M is tangent to the circle, K.

Hence we have five infinite sets of circles C_1, C_2, C_3, C_4, K tangent to five fixed lines, L_1, L_2, L_3, L_4, M. Since the circles, whatever the value of n, pass through the focus F and are always tangent to these five fixed lines, their centers describe five parabolas, P_1, P_2, P_3, P_4, Q whose vertices are at the mid-points of the perpendiculars from the focus to the five fixed lines. The feet of four of these perpendiculars are the vertices of the square formed by the four vertical tangents to the fixed parabolas and the center of this square is at the opposite extremity of the diameter of the circle K which has one extremity at the focus. Moreover, this center is on the line M and therefore, the line M is perpendicular to the diameter mentioned, i. e., the center of the square formed by the vertical tangents is at the foot of the perpendicular from the focus F to the line M. Hence the vertex of the parabola Q, which is the mid-point of this perpendicular, is the center of the square formed by the vertices of the four P-parabolas.

This would follow directly from the fact that the S-quadrangle in this case is formed by the four vertices of the G-parabolas and the vertices of the P-parabolas and the Q-parabola are the centers of the circles belonging to this S-quadrangle.

The results of this paper may be summed up as follows:

Given four confocal parabolas whose vertical tangents form a square;

THEOREM I. *Any four rectangular tangents to these four parabolas form a square whose vertices lie on four fixed lines and whose center lies on a fifth fixed line.*

THEOREM II. *Any two perpendicular lines through the common focus cut the four parabolas in four points which form an S-quadrangle, about which an infinite number of squares may be circumscribed whose vertices lie on four circles tangent to the four fixed lines of Theorem I, and whose centers lie on a fifth circle tangent to the fifth fixed line of Theorem I.*

THEOREM III. *The centers of the four circles in Theorem II lie on four parabolas confocal with the given parabolas, whose vertices are the mid-points of the sides of the quadrilateral formed by the vertices of the given parabolas, and therefore form a square. The center of this square is the vertex of a parabola, confocal with the others of the set, which is the locus of the center of the fifth circle in Theorem II.*

Hence we have nine confocal parabolas, five fixed lines, and five infinite sets of circles tangent to the five fixed lines and concurrent at the common focus of the parabolas. These seem to explain the apparent (though only apparent) dualism of the theorems relating to quadrilaterals having an infinite number of inscribed squares and quadrangles having an infinite number of circumscribed squares.*

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* For mention of this apparent dualism and doubt as to its nature, see the article by Thomas Clausen, l. c.